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Sharing Costs and the Compromise Solution.

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Sharing Costs and the Compromise Solution ^{*}

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Abstract

This paper introduces the **Compromise rule** for cost sharing problems. This rule fairly combines the equity and the growth-encouraging key principles. The way in which it is described also allows to endogenously identify the agents belonging to a “middle class” depending on how the costs of satisfying individual demands are distributed.

Keywords: Cost Sharing Rule, Local Progressiveness, Compromise Rule.

Journal of Economic Literature Classification Numbers: **C71, D63, D71.**

1 Introduction

Cost sharing is one of the oldest problems that faced the homo oeconomicus. Men and women accepted that their joint activity could be beneficial for all of them when collaborating. Nevertheless, a conflict has always emerged when deciding how the joint benefit from their collaboration should be distributed among them. The aim of this paper is to introduce a new procedure to share joint costs (or to share the benefits from cooperation) that can be understood as a compromise solution for this problem.

Consider a group of agents, each one wanting to develop some activity. Inherent to satisfying each agent's desires there is a cost, that we refer to as his stand-alone cost. Nevertheless, when all the agents operate together, the whole society must take responsibility for a joint cost below the aggregate of its members' stand-alone costs. The difference between the aggregate of the stand-alone costs and the joint cost can be interpreted as the social surplus. Therefore, associated to each cost sharing problem -i.e., how to distribute the joint cost among the agents- it emerges a surplus sharing problem, namely how to distribute the social surplus among the agents.

There is some social consensus on which solutions for cost sharing problems should be discarded, basically due to incentive-compatibility reasons;¹ or inspired in fairness principles; i.e., agents should be treated in a similar way. Nevertheless, there are some discrepancies among agents on how these 'equal treatment' principles should be interpreted. Some agents invoke that joint cost should be equally distributed among agents. In opposition to these agents, there are others claiming that the social surplus should be equally distributed. Clearly, these two opinions are confronted.

An even-handed way to address this dispute on uniformly distribute social surplus (or, similarly, joint cost) has been the adoption of the Proportional solution, inspired in the Aristotle's *proportional principle*² (Homans, 1974). It proposes that every agent's contribution to cover the joint cost determines the same part of his stand-alone cost. Moreover, it also proposes that every agent's reward from the social surplus determines the same part of his stand-alone cost. Nevertheless, the Proportional solution has been criticized when the agents' stand alone costs are highly unequal and the social surplus is either very high or very low, related to the stand-alone cost.

In this paper we introduce a new procedure to reduce the dispute between agents exhibiting a high stand-alone cost –who claim that join cost should be equally distributed– and those claiming that the social surplus should be equally distributed because their stand-alone cost is low. The solution that we propose is absent of some of the criticisms received

¹ To this respect, for a solution to be taken into consideration should satisfy that (a) no agent is imputed a share of the joint cost exceeding his stand-alone cost (individual rationality), and (b) no agent is imputed a share of the social surplus exceeding his stand-alone cost (non-negativity of the imputed cost).

² According Moulin (2002), this principle is described as “equals should be treated equally, and unequals, unequally in proportion to relevant similarities and differences.”

by the Proportional solution, mainly when the agents' stand-alone costs exhibit a high variance and either the social surplus or the joint cost are relatively small. Interesting enough, our proposal coincides with that of the Proportional solution in a wide family of problems, where the agents' stand-alone costs are not very different. Despite our problem exhibits a structure similar to the standard rationing problems (O'Neill, 1982), our treatment differs from the classical approach to these problems. Notice that the aggregate cost is very related to (and thus, not independent from) the individuals' stand alone costs. Is precisely such a relationship, as well as our interest on considering that aggregate costs and stand alone costs are highly correlated, that suggests not to consider throughout this paper the classical approach to rationing problems.

As previously mentioned, agents with a high stand-alone cost support the adoption of solutions equalizing the cost imputed to each agent. If, moreover, they are required to propose individually rational solutions, then these agents become supporters of the so-called Uniform-Cost solution.³ In contrast, agents with a low stand-alone cost always give support to solutions equalizing the social surplus attributed to each individual. If, moreover, the selected solutions need to satisfy non-negativity of the imputed cost, these agents support the adoption of the Uniform-Surplus solution.⁴ In the furtherance of reaching a compromise between these two categories of agents -the ones having a high stand-alone cost and these whose stand-alone cost is low- we design a solution fairly combining the Uniform-Cost and the Uniform-Surplus solution. The **Compromise solution** can be easily described as a convex combination of the Uniform-Cost and Uniform-Surplus solutions. To capture this idea of compromise between agents, the coefficient associated to the Uniform-Cost solution increases as the aggregate stand-alone cost grows; whereas it decreases as the social surplus raises. To do so, we resort to the **surplus ratio**, which describes a measure of the relative social surplus achieved throughout a given level of aggregate stand-alone cost.

In parallel to the above debate on which solution should be adopted, namely the Uniform-Cost and the Uniform-Surplus solution, there is another discussion whose arguments are established in terms of distributive fairness vs. incentive properties derived from the adopted solution. To this respect, and borrowed from the close literature on taxation (Young, 1988), we can mention that, by assuming that each agent's stand-alone cost is very related to his income, progressive solutions help to reduce social income inequality, whereas regressive solutions might enhance the possibilities of economical growth.⁵ Our point is that progressiveness and/or regressiveness should be considered as local properties rather than global concepts. To this respect, let us imagine that agents can be divided into two groups, the low-income agents, having a reduced stand-alone cost, and the high-

³ This solution is described in Subsection 2.2.

⁴ See Subsection 2.3 for a formal description.

⁵ Progressiveness entails that the proportion of the stand-alone cost that an agent is imputed as his share of the joint cost increases as his stand-alone-cost increases. A solution is regressive when this proportion decreases as the agent's stand-alone cost increases.

income agents whose stand-alone cost is high. When comparing how a solution treats any two agents, it is reasonable that the low-income agents (if any) find that the solution has behaved as progressive. But, when concentrating on high-income agents, each of them finds that the solution behaves as regressive. This property, which is particularly relevant when the surplus ratio is small, is satisfied by the Compromise solution (Proposition 1).

The above disquisition about low-income and high-income agents, related to their relative level of stand-alone cost, allows to endogenously describe a middle agent class whose members are considered by the high-income agents as belonging to the low-income category; whereas they are considered by low-income agents as high-income individuals.

As already mentioned, the Proportional solution is the most commonly adopted solution, and it is commonly considered a fair procedure to share costs. We also commented that the main criticisms to such a solution refer to the case where the agents' stand-alone costs are highly unequal, and thus the fulfillment of (local) regressiveness and/or progressiveness becomes relevant. Therefore, a comparison between the cost shares proposed by the Proportional and the Compromise solutions might shed some light on the extent at which the latter might constitute an improvement over the former. To this respect we find, Theorem 1, that these rules coincide when the agents' stand-alone costs do not exhibit a high variance. Moreover, our Theorem 2 states that when the surplus ratio is low, agents with low stand-alone cost prefer to adopt the Compromise rule rather than the application of the Proportional rule.

The rest of the paper is organized as follows. Section 2 introduces the model as well as the main definitions. The Compromise rule is introduced in Section 3, which also gathers the main results. Section 4 concludes. For exposition convenience, all the proofs and some auxiliary results have been relegated to the Appendix.

2 The Framework and Definitions

We consider a set $\mathcal{N} = \{1, \dots, i, \dots, n\}$ of agents, each of them needing a certain amount q_i of a perfectly divisible good. Given the agents' demands $Q = (q_1, \dots, q_i, \dots, q_n)$, we denote the aggregate demand as $q = \sum_{i=1}^n q_i$.

The production technology is summarized by the cost function $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which satisfies two natural properties. Monotonicity is a standard requirement, whereas sub-additivity guarantees the existence of a positive net benefit from cooperation.

- (a) **Monotonicity:** For any two quantities of good, say q and q' , if $q < q'$ then $C(q) \leq C(q')$.
- (b) **Sub-additivity:** For any two positive quantities of good, say q and q' ,

$$C(q + q') < C(q) + C(q').$$

A cost sharing problem is described through a pair (Q, C) . For notation simplicity, for a given problem (Q, C) we set its joint cost as $c = C(q)$. Similarly, the stand-alone cost of agent i is denoted by $c_i = C(q_i)$.

A **solution** for problem (Q, C) is a vector $x \in \mathbb{R}_+^n$ such that

$$\sum_{i=1}^n x_i = c.$$

Solution x for the problem (Q, C) satisfies the **stand-alone test** if no agent is imputed more than his stand-alone cost; i.e., for each agent i , $x_i \leq c_i$.

A **cost sharing rule**, or simply a **rule**, is a functional φ associating a solution to each problem (Q, C) . Therefore, given a problem (Q, C) , we interpret $\varphi_i(Q, C)$ as the cost that rule φ imputes to the agent i . Rule φ satisfies the stand-alone test if for each problem, say (Q, C) , and agent i , $x_i = \varphi_i(Q, C) \leq c_i$.

Associated with each problem (Q, C) , we can compute the social surplus that cooperation induces. It is described as

$$S(Q, C) = \sum_{i=1}^n c_i - c. \quad (1)$$

The **surplus ratio**, that measures the relative surplus that agents should share, is defined as⁶

$$\sigma(Q, C) = \frac{S(Q, C)}{\sum_{i=1}^n c_i}. \quad (2)$$

Notice that, since the cost function C is sub-additive, for each problem (Q, C) , $0 \leq \sigma(Q, C) \leq 1$. Since we are interested in rules (and thus solutions) fulfilling the stand-alone test, from now on we concentrate on problems (Q, C) such that $c_i > 0$ for each agent i .

Incentive-compatibility purposes yield to require cost sharing rules to be monotonic; i.e., for each problem (Q, C) and any two agents, say i and j , if $c_i \leq c_j$ then $\varphi_i(Q, C) \leq \varphi_j(Q, C)$.

Given a problem (Q, C) , a solution x for this problem is said **progressive** whenever for any two agents, i and j ,

$$c_i \leq c_j \text{ implies } \frac{x_i}{c_i} \leq \frac{x_j}{c_j}. \quad (3)$$

We say that rule φ is **progressive** if it associates a progressive solution to each problem (Q, C) . As opposition to progressiveness, solution for (Q, C) , say x , is **regressive** if for any

⁶ Note that when the stand-alone cost of each agent is zero, since C is sub-additive, the total cost to be shared is $0 \leq c \leq \sum_{i=1}^n c_i = 0$, and thus expression (1) is not well-defined. Therefore, for the sake of completeness, we convene that $\sigma(Q, C) = 1$ when $c_i = 0$ for each i .

two agents, i and j ,

$$c_i \leq c_j \text{ implies } \frac{x_i}{c_i} \geq \frac{x_j}{c_j}. \quad (4)$$

Rule φ is **regressive** whenever it associates a regressive solution to each problem (Q, C) .

The notions of regressiveness and progressiveness above are standard in the literature. In what follows we relax these requirements by proposing two weak notions for progressiveness (resp. regressiveness), not necessarily involving the whole society but a subset of agents. They are inspired on how agents are differentiated according to some specific problem. To fix ideas, let us consider that, given a problem (Q, C) , a group T of agents exhibit some similarities (or constitutes a category of agents). Their assessment on how fair a solution x is for this problem might depend on how the whole group perceive that this solution behaves for them, not taking into account the opinion of the agents outside this category.

Let us consider a problem (Q, C) , a solution x and a (non-empty) set of agents $T \subseteq \mathcal{N}$. We say that x is

- (a) **T-progressive** at (Q, C) , when condition (3) above holds for any two agents in T .
- (b) **T-regressive** at (Q, C) when condition (4) holds for any two agents in T .
- (c) **T^P-progressive** at (Q, C) , or perceived as progressive at the problem (Q, C) for agents in T , whenever for any two agents $i \in T$ and $j \in \mathcal{N}$

$$c_i \leq c_j \iff \frac{x_i}{c_i} \leq \frac{x_j}{c_j}.$$

- (d) **T^P-regressive** at (Q, C) , whenever for any two agents $i \in T$ and $j \in \mathcal{N}$

$$c_i \leq c_j \iff \frac{x_i}{c_i} \geq \frac{x_j}{c_j}.$$

Just to illustrate the difference between the above partial notions of progressiveness and regressiveness, let us consider a problem with 6 agents, whose stand-alone costs are $(c_i)_{i=1}^6 = (3, 4, 5, 90, 110, 130)$, the joint cost is $c = 186$, and consider solution $x = (1, 2, 3, 50, 60, 70)$. Note that this solution is neither progressive nor regressive. Now, consider the following sets of agents:

- (1) $T_1 = \{1, 2, 3\}$. Since

$$\frac{x_1}{c_1} = \frac{1}{3} < \frac{x_2}{c_2} = \frac{2}{4} < \frac{x_3}{c_3} = \frac{3}{5},$$

solution x is **T₁-progressive** at this problem.

(2) $T_2 = \{4, 5, 6\}$. Since

$$\frac{x_4}{c_4} = \frac{50}{90} > \frac{x_5}{c_5} = \frac{60}{110} > \frac{x_6}{c_6} = \frac{70}{130},$$

solution x is **T_2 -regressive** at this problem. Here, agents in T_2 can be interpreted as the ones amassing a huge proportion of the wealth. It is reasonable to consider that these agents do not take into account whether the solution treats the agents outside this group, when conforming their opinion about how appropriate solution x is.

(3) $T_3 = \{1, 2\}$. Since

$$\frac{x_1}{c_1} = \frac{1}{3} < \frac{x_2}{c_2} = \frac{2}{4} < \frac{x_j}{c_j} \text{ for all } j \notin R,$$

solution x is **T_3^P -progressive** at this problem. Note that all the agents in T_3 perceive that x behaves as a progressive solution. This is because they compare their imputed cost ratio with that of everybody else. This implies that, since the inequalities in the equation above are strict, no agent perceives that x behaves as a regressive solution when he compares to everybody else.

For a given cost sharing rule φ^ν , we define its imputed cost ratio for agent i at problem (Q, C) as the ratio

$$r_i^\nu(Q, C) = \frac{\varphi_i^\nu(Q, C)}{c_i}. \quad (5)$$

2.1 The Proportional Solution

As we have mentioned in the Introduction, proportionality is, in practice, the most commonly adopted solution concept. It requires that all the agents contribute to support the joint cost through the same proportion of their stand-alone costs. By extending this idea to any problem we can propose a formal definition of the Proportional rule.

Definition 1 The **Proportional rule** is the functional φ^P that associates to each problem (Q, C) and agent i the imputed cost

$$\varphi_i^P(Q, C) = \frac{c}{\sum_{j=1}^n c_j} c_i = [1 - \sigma(Q, C)] c_i. \quad (6)$$

Note that, for each given problem (Q, C) its proportional solution equalizes the imputed cost ratio for all the agents:

$$r_i^P(Q, C) = \frac{c}{\sum_{j=1}^n c_j} = [1 - \sigma(Q, C)] \text{ for each } i \in \mathcal{N}.$$

Therefore, since all the agents exhibit the same imputed cost ratio, conditions (3) and (4) are always fulfilled for any pair of agents; i.e., for each given problem, its proportional solution is both regressive and progressive.

2.2 The Uniform-Cost Solution

The Uniform-Cost rule associates to each problem an extremely regressive solution. It imputes each agent a cost so that the social surplus is mainly assigned to the agents with a highest stand-alone cost. Two restrictions are imposed. The first one is that no agent is imputed more than his stand-alone cost. The second one establishes an incentive-compatibility requirement: it should be monotonic.

Definition 2 The **Uniform-Cost rule** is the functional φ^{UC} associating to each problem (Q, C) and agent i the imputed cost $\varphi_i^{UC}(Q, C) = \min \{c_i, \alpha\}$, where α is the unique solution to

$$c = \sum_{j=1}^n \min \{c_j, \alpha\}. \quad (7)$$

2.3 The Uniform-Surplus Solution

As opposition to the highly regressive solution described in Subsection 2.2, we now deal with a procedure to select, for each given problem, an extremely progressive solution. In this case, agents share equally the social surplus. Recall that, by definition of a solution, each agent should be imputed a non-negative share of the cost. This determines the maximal surplus that each agent might be imputed, not exceeding his stand-alone cost. The rule that systematically selects this solution for each given problem is called the Uniform-Surplus rule. The duality between the Uniform-Cost and the Uniform-Surplus rules allows to asseverate that, since the former is a regressive rule the latter must satisfy progressiveness.

Definition 3 The **Uniform-Surplus rule** is the functional φ^{US} associating to each problem (Q, C) and agent i the imputed cost $\varphi_i^{US}(Q, C) = \max \{0, c_i - \beta\}$, where β is the unique solution to

$$c = \sum_{j=1}^n \max \{0, c_j - \beta\}. \quad (8)$$

2.4 An Example to Summarize

Let us consider a helpful example to illustrate how the solutions above can be computed. It allows to introduce a comparative of the solutions in terms of fairness. We consider a

four-agent society, $\mathcal{N} = \{1, 2, 3, 4\}$ facing the cost function $C(q) = \sqrt{q}$. Let us assume that the vector of agents' demands is $Q = (64, 100, 1296, 4624)$. Therefore, the aggregate demand is $q = 6084$, and thus the joint cost is $c = 78$, while the agents' stand-alone costs are $(c_i)_{i=1}^4 = (8, 10, 36, 68)$, which implies that the social surplus is $S(Q, C) = 44$ and the surplus ratio is $\sigma(Q, C) \approx 0.36$.

Let us now concentrate on computing, for this problem, the three solutions above mentioned.

- (a) **The Proportional solution.** Since $\sigma(Q, C) \approx 0.36$, by applying the formula in expression (6) we have that

$$\varphi^P(Q, C) \approx (5.11, 6.39, 23.02, 43.48).$$

- (b) **The Uniform-Cost solution.** When solving equation (7) we have that $\alpha = 30$. Therefore,

$$\varphi^{UC}(Q, C) = (8, 10, 30, 30).$$

- (c) **The Uniform-Surplus solution.** When solving equation (8) we have that $\beta = 12$. Therefore,

$$\varphi^{US}(Q, C) = (0, 0, 23, 55).$$

It is remarkable that agents 1 and 2 find that the solution harms their interests, when the Uniform-Cost solution is adopted. This is because no (positive) share of the social surplus is attributed to them. Relative to the Uniform-Surplus solution, agents 3 and 4 might claim that it is an unfair solution because agents 1 and 2 do not contribute to cover the joint cost, whereas their individual demand is costly. Moreover, these agents might also claim that the adoption of the Uniform-Surplus rule incentive the agents with low demand to inflate it. To illustrate this situation, let us assume that agent 1 tries to maximize some utility function, increasing in the amount of good he consumes and decreasing in the cost he is imputed. If agents other than 1 select the same quantity as in this example, whereas 1 is free to select any quantity he wishes, say q_1 , the surplus to be distributed among agents is

$$S(Q(q_1), C) = 114 + \sqrt{q_1} - \sqrt{6020 + q_1},$$

where $Q(q_1)$ denotes the vector of demands $(q_1, 100, 1296, 4624)$. Now consider that the joint cost is distributed as the Uniform-Surplus solution prescribes. Simple computations yield to see that $\varphi_1^{US}(Q(q_1), C) = 0$ for $q_1 \lesssim 161.04$, which implies that agent 1 exhibits incentives to raise his demand.

For the Proportional solution there is a traditional criticism invoking to solidarity principles that should contribute to reduce rooted inequalities. Note that the demands exerted

by agents 1 and 2 are very low, related to that of 3 and 4. It can be explained in terms of wealth inequality among these two groups of agents. Therefore, whenever that promoting a reduction of the wealth inequality is a relevant objective, it is important to associate a significant part of the social surplus to the less favored agents. Unfortunately, this objective might not be reached by resorting to the Proportional solution. Therefore, there is a trade-off between a “solidarity” principle, requiring that the agents with lower demand should perceive that solutions are progressive, and an incentive-compatibility requirement which limits the level of progressiveness that solutions might exhibit.

3 The Compromise Solution

In this section we introduce the **Compromise Solution**, and develop an analysis of its behavior in term of (partial) regressiveness/progressiveness.

To illustrate how the Compromise rule operates, we reconsider the Example explored in Subsection 2.4, and consider some different cost functions. For, we consider the (fixed) demands vector $Q = (64, 100, 1296, 4624)$, and a family of cost functions we index as C_δ , with $C_\delta(q) = q^\delta$. Here, $\delta \in (0, 1)$ is a parameter indicating the economies of scale associated to the available technology. Note that, for δ given, the joint cost that the set of agents face is $C_\delta(q) = 6084^\delta$, and $S(Q, C_\delta) = 64^\delta + 100^\delta + 1296^\delta + 4624^\delta - 6084^\delta$. Now, consider two ‘extreme’ cases for δ ,

- (H) $\delta = 0.95$. For such a high value of δ we have that the surplus ratio $\sigma(Q, C_\delta) = 0.033$. Now, consider a solution for such a problem, say x . Since the surplus ratio is very low, the closer to the Uniform-Cost solution x is, the more unmet the solidarity principle is. Moreover, provided that the surplus ratio is low, for x close enough to the Uniform-Surplus solution, the potential incentive-compatibility problem vanishes.
- (L) $\delta = 0.005$. For such a low value of δ we have that the surplus ratio $\sigma(Q, C_\delta) = 0.7467$. Now, consider a solution for such a problem, say x . Since the surplus ratio is high, when x is close to the Uniform-Surplus an incentive-compatibility problem might emerge. Nevertheless, for x relatively close to the Uniform-Cost solution, the potential incentive-compatibility problem is avoided without (necessarily) incurring in a lack of solidarity.

What the above situations illustrate is that an accurate combination of the Uniform-Cost and the Uniform-Surplus solutions can help to reduce the trade-off between solidarity and incentive compatibility. This is gathered up by the Compromise solution, as captured by the expression (9) below. It associates weights to the two ‘uniform’ solutions, so to capture the ideas reflected in the two situations above, and thus derived from the surplus ratio associated to each specific problem.

Definition 4 The **Compromise rule** is the functional $\varphi^{\mathcal{K}}$ that associates to each cost sharing problem (Q, C) the solution

$$\varphi^{\mathcal{K}}(Q, C) = \sigma(Q, C) \varphi^{UC}(Q, C) + [1 - \sigma(Q, C)] \varphi^{US}(Q, C). \quad (9)$$

To illustrate how the Compromise rule operates, consider the problem in Subsection 2.4 above; i.e., $Q = (64, 100, 1296, 4624)$ and $C(q) = \sqrt{q}$. Therefore, $\sigma(Q, C) \approx 0.36$, and thus

$$\varphi^{\mathcal{K}}(Q, C) \approx 0.36 \varphi^{UC}(Q, C) + 0.64 \varphi^{US}(Q, C) \approx (2.89, 3.61, 25.52, 45.98).$$

The following table helps to compare the agents' imputed cost ratios for this problem, associated to each of the four solution concepts: The Uniform-Cost, the Uniform-Surplus, the Proportional, and the Compromise solutions.

$r_i^{\tau}(Q, C)$ (in %)	Agent 1	Agent 2	Agent 3	Agent 4
Uniform-Cost solution	100.00	100.00	83.33	44.12
Uniform-Surplus solution	0.00	0.00	63.89	80.88
Proportional solution	63.93	63.93	63.93	63.93
Compromise solution	36.07	36.07	70.90	67.62

Related to the table above, let us consider the sets of agents $S = \{1, 2, 3\}$ and $T = \{3, 4\}$, and let x be the Compromise solution for problem $(Q, C) = ((64, 100, 1296, 4624), C(q) = \sqrt{q})$. We have that x is S^P -progressive at (Q, C) -and thus it is S -progressive at (Q, C) - whereas it is T -regressive at (Q, C) . Note that $\{3\} = S \cap T$ is compound by the agents exercising a high demand among the ones with low demand and, at the same time, the agents with a low demand among the ones whose demand is high. This is why we refer this set as the 'middle class' agents.

As Proposition 1 below establishes the observation above occurs when the Compromise solution is adopted in problems with a low surplus ratio: There is a 'lower class' that perceives that the solution is progressive, and thus it is adequate to reach a wealth redistribution; and there is a, 'upper class' that perceives that the solution is regressive, so it induce right incentives from an economic growth perspective.

Proposition 1 Let (Q, C) be a problem with $\sigma(Q, C) < 0.5$, and $x = \varphi^{\mathcal{K}}(Q, C)$ be its Compromise solution. Then, there are two sets of agents $\mathcal{N}'$ and $\mathcal{N}''$, with $\mathcal{N} = \mathcal{N}' \cup \mathcal{N}''$ such that

- (a) x is $\mathcal{N}'$ -progressive at (Q, C) ;
- (b) x is $(\mathcal{N}'' \setminus \mathcal{N}')$ -regressive at (Q, C) ;

- (c) if agent i is in $\mathcal{N}'$ and, for agent j , $c_j \leq c_i$, then $j \in \mathcal{N}'$;
- (d) if agent i is in $\mathcal{N}''$ and, for agent j , $c_j \geq c_i$, then $j \in \mathcal{N}''$; and
- (e) for any two agents $i \in \mathcal{N}''$ and $j \in \mathcal{N}' \setminus \mathcal{N}''$

$$r_i^{\mathcal{K}}(Q, C) \geq r_i^P(Q, C) = r_j^P(Q, C) \geq r_j^{\mathcal{K}}(Q, C).$$

For completeness purposes we introduce the next result informing that the conclusions in Proposition 1 above are reversed for high values of the surplus ratio.

Proposition 2 Let (Q, C) be a problem with $\sigma(Q, C) > 0.5$, and let $x = \varphi^{\mathcal{K}}(Q, C)$ be its Compromise solution. Then, there are two sets of agents $\mathcal{N}'$ and $\mathcal{N}''$, with $\mathcal{N} = \mathcal{N}' \cup \mathcal{N}''$ such that

- (a) x is $\mathcal{N}'$ -regressive at (Q, C) ;
- (b) x is $(\mathcal{N}'' \setminus \mathcal{N}')$ -progressive at (Q, C) ;
- (c) if agent i is in $\mathcal{N}'$ and, for agent j , $c_j \leq c_i$, then $j \in \mathcal{N}'$;
- (d) if agent i is in $\mathcal{N}''$ and, for agent j , $c_j \geq c_i$, then $j \in \mathcal{N}''$; and
- (e) for any two agents $i \in \mathcal{N}''$ and $j \in \mathcal{N}' \setminus \mathcal{N}''$

$$r_i^{\mathcal{K}}(Q, C) \leq r_i^P(Q, C) = r_j^P(Q, C) \leq r_j^{\mathcal{K}}(Q, C).$$

Figure 1 below illustrates how the degree of sub-additivity of the cost function affects the imputed cost ratio associated to the agents. Here we have considered the agents' demand $Q = (64, 100, 1296, 4624)$. We studied the imputed cost ratio for agents 1 and 4 when the cost function adopts the shape $C_\delta(q) = q^\delta$. Note that, for Q given, the social surplus decreases with δ . This figure illustrates that for high values of δ ($\delta > 0.3282$, which corresponds to the case that the surplus ratio is 0.5) agent 1 supports the adoption of the Compromise rule rather than the Proportional rule. For δ low enough ($\delta \leq 0.2436$ in this instance) all the agents are indifferent between adopting the Proportional or the Compromise rules. There is a small interval where the agent with the lowest demand prefers to adopt the Proportional instead on the Compromise rule. When concentrating on the agent exerting the highest demand, the comparisons of the Proportional and the Compromise rule are, in this instance, negligible.

Figure 1: Sub-additivity and the Imputed Cost Ratio

Figure 2 below illustrates how the imputed cost ratio for an agent varies as his demand does. For, we consider four-agent problems that face the cost functions $C_t(q) = q^{t/4}$, $t = 1, 2, 3$, and the demand vectors are $Q(q_1) = (q_1, 250, 360, 490)$. The figure illustrates that when q_1 is relatively small, agent 1's imputed cost ratio under the Compromise solution is lower than that provided by the Proportional solution. This inequality reverses when q_1 is large enough. It is important to stress that there is an interval of values for q_i where the Compromise and the Proportional solutions coincide (and hence their respective imputed cost ratios).

Figure 2: Sub-additivity and the Imputed Cost Ratio

To be precise, when the demands by agents other than 1 are $Q_{-1} = (250, 360, 490)$,

(1) when $C(q) = C_1(q) = \sqrt[4]{q}$,

(L) for $q_1 < 33.815$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_1) < \varphi_1^P(Q(q_1), C_1)$;

(H) for $q_1 > 62900$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_1) > \varphi_1^P(Q(q_1), C_1)$; and

(M) for $33.815 \leq q_1 \leq 62900$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_1) = \varphi_1^P(Q(q_1), C_1)$;

(2) when $C(q) = C_2(q) = \sqrt{q}$,

(L) for $q_1 < 73.333$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_2) < \varphi_1^P(Q(q_1), C_2)$;

(H) for $q_1 > 2900$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_2) > \varphi_1^P(Q(q_1), C_2)$; and

(M) for $73.333 \leq q_1 \leq 2900$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_2) = \varphi_1^P(Q(q_1), C_2)$;

(3) when $C(q) = C_3(q) = \sqrt[4]{q^3}$,

(L) for $q_1 < 205.62$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_3) < \varphi_1^P(Q(q_1), C_3)$;

(H) for $q_1 > 487.4$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_3) > \varphi_1^P(Q(q_1), C_3)$; and

(M) for $205.62 \leq q_1 \leq 487.4$, $\varphi_1^{\mathcal{K}}(Q(q_1), C_3) = \varphi_1^P(Q(q_1), C_3)$.

By way of a summary, for a given problem (Q, C) , and any agent i , the difference $|\varphi_i^{\mathcal{K}}(C, Q) - \varphi_i^P(C, Q)|$ grows on the standard deviation of the agents' demands, whereas it is decreasing on the degree of sub-additivity exhibited by the cost function. This is reflected in Proposition 3 below, which introduces a sufficient (and almost necessary) condition under which the Compromise and Proportional solution for a given problem coincide.

Proposition 3 Let (Q, C) a problem satisfying that, for each $i \in \mathcal{N}$,

$$\max\{S(Q, C), c\} \leq nc_i, \quad (10)$$

then $\varphi^{\mathcal{K}}(q, C) = \varphi^P(q, C)$.

There are two sources of comparison which might introduce substantive differences among agents for a specific problem. The first one is the **per capita joint cost**, whereas the second one is the **per capita surplus**. Any agent whose stand alone cost is between these two values should be an unequivocal supporter of one of the two extreme rules. Note that, for a given problem, say (Q, C) an agent i would support the Uniform-Cost solution whenever

$$\frac{S(Q, C)}{n} < c_i < \frac{c}{n}. \quad (11)$$

This is because, in i 's view, the surplus level is reduced. Therefore, he would like agents' imputed costs to be equalized. In contrast, when for agent i

$$\frac{S(Q, C)}{n} > c_i > \frac{c}{n}, \quad (12)$$

his opinion will be that surplus is high and thus he will claim that are the agents attributed surplus which have to be equalized. Associated to any given problem, we can then distinguish three categories of agents: the ones for which condition (11) is satisfied, those for which condition (12) holds, and the agents for which none of these conditions is fulfilled. Is precisely when all the agents belong to the last category, and thus their opinion on which extreme solution is not clearly extreme, that we can interpret that the agents' demands seem similar. Proposition 3 above points out that under such a similarity, the Compromise and the Proportional rules coincide.

Proposition 4 below establishes a sufficient condition for the Compromise and Proportional solutions coincide for a given problem.

Proposition 4 Let (Q, C) be a problem such that $c = S(Q, C)$, then for each agent i

$$\varphi_i^{\mathcal{K}}(Q, C) = \varphi_i^P(Q, C) = \frac{1}{2}c_i.$$

As we now report, the conditions in Propositions 3 and 4 are also necessary to guarantee the coincidence of the Proportional and the Compromise rules.

Theorem 1 For a given problem (Q, C) , $\varphi^{\mathcal{K}}(Q, C) = \varphi^P(Q, C)$ if, and only if, either

- (a) $c = S(Q, C)$, or
- (b) for each $i \in \mathcal{N}$,

$$\max\{S(Q, C), c\} \leq nc_i.$$

Our last result establishes that agents exhibiting a low stand-alone cost are not hurt when the Compromise rule is adopted instead of the Proportional rule.

Theorem 2 Let (Q, C) be a problem such that $S(Q, C) < c$. For each $i \in \mathcal{N}$ such that $nc_i < S(Q, C)$, $\varphi_i^{\mathcal{K}}(Q, C) < \varphi_i^P(Q, C)$.

4 Final Remarks

Our model in Section 2 can be easily adapted to some frameworks where either agents' consumption does not correspond to one homogeneous (private) good⁷ or the dilemma progressiveness vs. regressiveness plays a central role in the selection of a specific rule (as in some rationing situation such as the bankruptcy problem (O'Neill, 1982), or taxation systems). In this section we explore how to adapt the Compromise rule to these close frameworks.

4.1 Demand for Bundles

Let us imagine that the agents are public hospitals. Each hospital needs to stock up on several basic goods (bandage, disinfectants, alcohol, etc.) When these hospitals commit to resort to a common central purchasing entity they can find some market advantage which turn out in a common saving (which does not appears when each hospital operates in isolation). In such a case it might be difficult to give an easy, direct description of how much each hospital should contribute to the overall joint cost. Notice that when, for instance, the provider who delivers alcohol might also deliver some disinfectants, it might reduce the (unitary) price of the alcohol for guaranteeing to be the provider for disinfectants.

An aspect that should be also observed by the distribution of the joint cost among the hospitals become from a solidarity perspective. It can be the case that Hospital A exhibit a huge demand because it is a big hospital operating in a crowded city, whereas Hospital B operates in a sparsely populated rural area. Notice that Hospital A has a huge budget (because the number of potential patients uses to be a relevant variable when determining each hospital budget). Moreover, a relevant part of the common savings can be attributed to the large demand of Hospital A. Therefore, at mid term, Hospital B is condemned to disappear because its lowering budget is incompatible with a raising provision cost. So, it is important to appeal to solidarity principles allowing to reduce the imputed cost to the agents exerting an individual demand whose cost is low (as illustrated through Hospital B).

4.2 Bankruptcy Problems

In a bankruptcy problem there is an agent who owe some amount of money to n different creditors. The debt to creditor i is denoted by d_i . Nevertheless the assets of the debtor are valued E , $E < \sum_i d_i$. The question is how to distribute E among the creditors so that none of them is assigned more than his credit.

Note that in this model, as introduced by O'Neill (1982), a solution describes how much each creditor recovers. Therefore the interpretations provided in Section 2 has to be con-

⁷ Think of the case where either there is a public good and partial exclusion on its consumption might be exerted, or the agents consumption is described thought a bundle of $k > 1$ goods, ...

sidered as dual to the ones in this subsection. In particular, in the cost-sharing framework agents like to pay as few as possible; whereas for bankruptcy problems, agents prefer to be reimbursed as much as possible. Therefore, redistribution aims induce that, from the perspective of low-debtors, it is desirable solutions to be regressive. Additionally, coalitional cohesion implies that solutions have to be insensitive to huge debts (Curiel et al., 1987). Having all these ingredients in mind, the Compromise rule introduced in Section 3 can be adapted to the bankruptcy problems scenario as follows.

For a given problem (d, E) , where $d \in \mathbb{R}_+^n$ is the vector describing the agents' credits, let us define

(EA) its Equal Awards solution, $\psi^{EA}(d, E)$, as the vector $y \in \mathbb{R}_+^n$ such that agent i receives $y_i = \min\{d_i, e\}$, being e the unique solution to

$$E = \sum_{j=1}^n \min\{d_j, e\}. \quad (13)$$

(REL) its Restricted Equal Losses solution,⁸ $\psi^{REL}(d, E)$, as the vector $y \in \mathbb{R}_+^n$ such that agent i receives $y_i = \max\{0, \hat{d}_i - \ell\}$, where for each i , $\hat{d}_i = \min\{d_i, E\}$, and ℓ the unique solution to

$$E = \sum_{j=1}^n \max\{0, \hat{d}_j - \ell\}. \quad (14)$$

(RBR) its Restricted Bankruptcy Ratio as the value

$$b(d, E) = \frac{\sum_{j=1}^n \hat{d}_j - E}{\sum_{j=1}^n \hat{d}_j} \quad (15)$$

For the case of bankruptcy problems, we define the **Consensus rule** as the function $\psi^{\mathcal{K}}$ which associates to each problem (d, E) the vector of shares

$$\psi^{\mathcal{K}}(d, E) = b(d, E) \psi^{EA}(d, E) + (1 - b(d, E)) \psi^{REL}(d, E) \quad (16)$$

4.3 Personal Income Taxation

Personal income tax systems have been deeply explored. One of the main reasons is that there are several relevant economic questions whose answer is much dependent on how personal income is taxed: Rent distribution, employment, economic growth, ...

⁸ Note that the Restricted Equal Losses solution differs from the well-known Constrained Equal Losses solution because in the former the creditors' claims are truncated (by the value of debtor's assets), whereas it is not the case for the latter. Clearly, the two solutions coincide for any problem (d, E) where $\max d_j \leq E$.

A simple way to illustrate how individual income should be taxed comes from the following sequential process. The government decides a (personal income) tax system, namely a function $\tau: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$, assigning each vector of agents' incomes $\omega = (\omega_i)_{i=1}^n$ the contribution of each agent to the overall tax collection. Therefore, for ω given, agent i 's contribution is $\tau_i(\omega)$. Note that τ should fulfill some standard properties, among them (1) feasibility: for each before-taxes income vector ω and agent i , $0 \leq \tau_i(\omega) \leq \omega_i$, and (2) cross-monotonicity: for each ω and any two agents i and j , if $\omega_i \leq \omega_j$ then $\tau_i(\omega) \leq \tau_j(\omega)$. Once a tax system is decided, agents decide their own effort (and thus their individual before-tax income) so to maximize their utility level. It is reasonable to assume that agent's utility is increasing in his after-tax income and decreasing in the exerted effort. The government's target might be described through a series of partial objectives, among them we can mention (1) to collect at least a given quantity T (in other words, provided that agents' incomes are summarized by ω , $\sum_{i=1}^n \tau_i(\omega) \geq T$); (2) to promote the maximum aggregate after-tax income; i.e., to maximize $\sum_{i=1}^n (\omega_i - \tau_i(\omega))$; and (3) to reduce, as much as possible, the inequality of agents' after-tax incomes. Note that partial objectives (1) and (2) above implies that $\sum_{i=1}^n \tau_i(\omega) = T$. Condition (2) invokes that agents should have correct incentives to exert a high effort level, whereas Condition (3) requires the tax system to be progressive. Therefore, it is still important to tackle a solidarity vs. incentive compatibility trade-off. This can be done through the adoption of the Compensatory Tax system below, that exhibits some similarities with the Compromise rule introduced in Section 3.

Let us consider a collection target T , and assume that agents' incomes are summarized through vector ω .⁹ We define

(HT) the Head Tax solution for problem (ω, T) as the vector $z = \tau^H(\omega, T)$ associating each agent the tax $z_i = \min\{\omega_i, \lambda\}$, where λ is the unique solution to

$$T = \sum_{j=1}^n \min\{\omega_j, \lambda\}. \quad (17)$$

(LT) the Leveling Tax solution for problem (ω, T) as the vector $z = \tau^L(\omega, T)$ associating each agent the tax $z_i = \max\{0, \omega_i - \mu\}$, where μ is the unique solution to

$$T = \sum_{j=1}^n \max\{0, \omega_j - \mu\}. \quad (18)$$

⁹ In what follows we assume that the level of agents' income is given. The analysis of the system as a whole, where agents' decisions on effort is influenced by the selection of a specific tax system, keeps for future research. Moreover, we introduce an additional abuse of notation which we hope do not induce misunderstandings. Since tax rules outputs depend not only in the agents' incomes, ω , but also in the collection target T , we denote by $\tau(\omega, T)$ the vector describing the tax imputed to each agent that allows to collect T .

(TB) the Tax Burden Degree of problem (ω, T) as the ratio

$$\mathcal{D}(\omega, T) = \frac{T}{\sum_{j=1}^n \omega_j} \quad (19)$$

We define the **Compensatory Tax rule** as the function $\tau^{\mathcal{K}}$ that associates each problem (ω, T) the vector of taxes

$$\tau^{\mathcal{K}}(\omega, T) = \mathcal{D}(\omega, T) \tau^H(\omega, T) + (1 - \mathcal{D}(\omega, T)) \tau^L(\omega, T). \quad (20)$$

Note that the Compensatory Tax rule, as above described, might propose tax schemes exhibiting some problems. Nevertheless, as we argue below, real-life considerations point out that these theoretical problems never occur in practice.

- (a) The tax scheme might be regressive from the low-income agents' perspective. This can be the case where Proposition 2 applies. Notice that this result holds when the collection target T is high enough (it must exceed half of the aggregate income). Nevertheless, when analyzing the OECD and Eurostat records, it is realized that tax revenue rates are not so high.¹⁰
- (b) The tax system is not income-monotonic. The Compensatory Tax rule, as it is described, might be decreasing in some agent income. Nevertheless, this can only be the case where the income obtained by this agent exceeds the collection target T ,¹¹ which is not a very realistic situation.

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¹⁰ Eurostat reports that “In 2015, tax revenue (including social contributions) in the EU-28 stood at 40.0 % of GDP”. In the same report says that “the ratio of 2015 tax revenue to GDP was highest in France (47.9 % of GDP), Denmark (47.6 % of GDP) and Belgium (47.5 % of GDP); the lowest shares were recorded in Ireland (24.4 % of GDP), Romania (28.0 % of GDP), Bulgaria (29.0 %), Lithuania (29.4 %) and Latvia (29.5 %) as well as Switzerland (28.1 %).” See http://ec.europa.eu/eurostat/statistics-explained/index.php/Tax_revenue_statistics, accessed on September 18, 2017.

¹¹ The Consensus rule, in Subsection 4.2, do not exhibit this problem because in its description we resort to the Restricted Equal Losses solution.

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APPENDIX

We now concentrate on proving the results in the paper. In what follows we consider a given problem (Q, C) , so that it is perfectly understood that α is the parameter obtained by solving equation (7) for problem (Q, C) ; β is obtained by solving equation (8) for our given problem; and $\sigma = \sigma(Q, C)$ is the surplus ratio associated with this problem.

For simplicity of exposition we assume that agents are labeled according their stand-alone cost; i.e. $i < j$ implies $c_i \leq c_j$. This allows to identify agents i_α and i_β so that (a) $\varphi_i^{UC}(Q, C) = \alpha$ if, and only if, $i \geq i_\alpha$; and (b) $\varphi_i^{US}(Q, C) = 0$ if, and only if, $i \leq i_\beta$.

We now propose an alternative, useful piece-wise expression for the Compromise rule.

$$\varphi_i^{\mathcal{K}}(Q, C) = \begin{cases} \alpha\sigma + (c_i - \beta)(1 - \sigma) & \text{if } \max\{\alpha, \beta\} < c_i \\ \alpha\sigma & \text{if } \alpha \leq c_i \leq \beta \\ c_i - \beta(1 - \sigma) & \text{if } \beta \leq c_i \leq \alpha \\ c_i\sigma & \text{if } c_i < \min\{\alpha, \beta\} \end{cases} \quad (21)$$

A Proof of Proposition 1

Assume that $\sigma < 0.5$. Therefore, by Alcalde and Peris (2017, Theorem 1), $\alpha > \beta$ and $i_\alpha \geq i_\beta$. Hence, equation (21) above reduces to

$$\varphi_i^{\mathcal{K}}(Q, C) = \begin{cases} \alpha\sigma + (c_i - \beta)(1 - \sigma) & \text{if } \beta < \alpha < c_i \\ c_i - \beta(1 - \sigma) & \text{if } \beta \leq c_i \leq \alpha \\ c_i\sigma & \text{if } c_i < \beta < \alpha \end{cases}$$

The following lemma relates the parameters α and β with the valuation of σ . Let us define for s and t in $\{1, \dots, n-1\}$,

$$H(s, t) = \frac{\sigma}{n-s} \left(c - \sum_{j=1}^s c_j \right) - \frac{1-\sigma}{n-t} \left(\sum_{j=t+1}^n c_j - c \right).$$

Lemma 1 Assume that $\sigma < 0.5$. Then, $H(i_\alpha, i_\beta) \geq H(i_\beta, i_\beta) > 0$.

Proof. Let us compute $H(i_\beta, i_\beta)$.

$$\begin{aligned} H(i_\beta, i_\beta) &= \frac{\sigma}{n-i_\beta} \left(c - \sum_{j=1}^{i_\beta} c_j \right) - \frac{1-\sigma}{n-i_\beta} \left(\sum_{j=i_\beta+1}^n c_j - c \right) \\ &= \frac{1}{n-i_\beta} \left(c - \sigma \sum_{j=1}^{i_\beta} c_j - (1-\sigma) \sum_{j=i_\beta+1}^n c_j \right). \end{aligned}$$

Then, $H(i_\beta, i_\beta) > 0$. This is because $2c > \sum_{j=1}^n c_j$, and

$$\begin{aligned} c - \sigma \sum_{j=1}^{i_\beta} c_j - (1-\sigma) \sum_{j=i_\beta+1}^n c_j &= c - \frac{\sum_{j=1}^n c_j - c}{\sum_{j=1}^n c_j} \sum_{j=1}^{i_\beta} c_j - \frac{c}{\sum_{j=1}^n c_j} \sum_{j=i_\beta+1}^n c_j = \\ &= \frac{2c - \sum_{j=1}^n c_j}{\sum_{j=1}^n c_j}. \end{aligned}$$

Now, we show that $H(i_\beta, i_\beta) \leq H(i_\beta + 1, i_\beta)$.

$$H(i_\beta + 1, i_\beta) = \frac{\sigma}{n-(i_\beta + 1)} \left(c - \sum_{j=1}^{i_\beta+1} c_j \right) - \frac{1-\sigma}{n-i_\beta} \left(\sum_{j=i_\beta+1}^n c_j - c \right).$$

Therefore,

$$\begin{aligned} H(i_\beta, i_\beta) &\leq H(i_\beta + 1, i_\beta) \iff \\ &\iff \frac{c - \sum_{j=1}^{i_\beta} c_j}{n-i_\beta} \leq \frac{c - \sum_{j=1}^{i_\beta+1} c_j}{n-(i_\beta + 1)} \iff \\ &\iff (n-i_\beta-1) \left(c - \sum_{j=1}^{i_\beta} c_j \right) \leq (n-i_\beta) \left(c - \sum_{j=1}^{i_\beta+1} c_j \right) \iff \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow -(n - i_\beta - 1) \sum_{j=1}^{i_\beta} c_j + (n - i_\beta) \sum_{j=1}^{i_\beta+1} c_j \leq c \quad \Leftrightarrow \\
&\Leftrightarrow (n - i_\beta) c_{i_\beta+1} \leq c - \sum_{j=1}^{i_\beta} c_j \quad \Leftrightarrow \\
&\Leftrightarrow c_{i_\beta+1} \leq \frac{c - \sum_{j=1}^{i_\beta} c_j}{n - i_\beta},
\end{aligned}$$

which is true by definition of i_β . Finally, since $i_\alpha \geq i_\beta$, by iteratively applying this reasoning, we obtain $H(i_\beta, i_\beta) \leq H(i_\alpha, i_\beta)$. ■

As an immediate consequence, taking into account that

$$\alpha = \frac{c - \sum_{j=1}^{i_\alpha} c_j}{n - i_\alpha}, \text{ and } \beta = \frac{\sum_{j=i_\beta+1}^n c_j - c}{n - i_\beta},$$

we have that, whenever $\sigma < 0.5$,

$$\alpha\sigma - \beta(1 - \sigma) > 0. \tag{22}$$

Now, let us define

$$\begin{aligned}
\mathcal{N}' &= \{i \in \mathcal{N} \text{ such that } c_i \leq \alpha\}, \text{ and} \\
\mathcal{N}'' &= \left\{i \in \mathcal{N} \text{ such that } c_i > \frac{1 - \sigma}{\sigma} \beta\right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{N}' \setminus \mathcal{N}'' &= \left\{i \in \mathcal{N} \text{ such that } c_i \leq \frac{1 - \sigma}{\sigma} \beta\right\} \\
\mathcal{N}'' \setminus \mathcal{N}' &= \{i \in \mathcal{N} \text{ such that } c_i > \alpha\}.
\end{aligned}$$

Since $\alpha\sigma > \beta(1 - \sigma)$, then $\mathcal{N} = \mathcal{N}' \cup \mathcal{N}''$. Moreover,

- (i) if $i \in \mathcal{N}'$ and $c_j \leq c_i$ then $j \in \mathcal{N}'$, as established in statement (c) of Proposition 1; and
- (ii) if $i \in \mathcal{N}''$ and $c_j \geq c_i$ then $j \in \mathcal{N}''$, as established in statement (d) of Proposition 1.

We analyze now the relationship between the imputed cost ratios associated to the Compromise and the Proportional rules for each agent i in $\mathcal{N}' \setminus \mathcal{N}''$. Consider the following two cases, that exhaust all the possibilities.

(a) $\beta \leq c_i \leq \alpha$. Then, by (21),

$$\begin{aligned}\varphi_i^{\mathcal{K}}(Q, C) &= c_i - \beta(1 - \sigma) \leq c_i - \frac{1 - \sigma}{n} \left(\sum_{j=1}^n c_j - c \right) = c_i - \frac{c}{\sum_{j=1}^n c_j} \frac{\sum_{j=1}^n c_j - c}{n} = \\ &= c_i - \frac{c}{n} \frac{\sum_{j=1}^n c_j - c}{\sum_{j=1}^n c_j} = c_i - \frac{c}{n} \sigma \leq c_i - \sigma c_i = (1 - \sigma) c_i.\end{aligned}$$

Therefore, by equation (22),

$$r_i^{\mathcal{K}}(Q, C) = \frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} \leq 1 - \sigma = \frac{\varphi_i^P(Q, C)}{c_i} = r_i^P(Q, C). \quad (23)$$

(b) $c_i \leq \min\{\alpha, \beta\}$. Then, by (21),

$$\frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} = \sigma < 1 - \sigma = \frac{\varphi_i^P(Q, C)}{c_i}.$$

So, for each $i \in \mathcal{N}' \setminus \mathcal{N}''$,

$$r_i^{\mathcal{K}}(Q, C) \leq r_i^P(Q, C). \quad (24)$$

Now, to complete the proof of assertion (e) in Proposition 1, let us consider agent $i \in \mathcal{N}''$, then

$$c_i > \frac{1 - \sigma}{\sigma} \beta > \beta.$$

Let us consider the following two cases, which exhaust all the possibilities.

(1) $\beta < \alpha < c_i$. Then,

$$\begin{aligned}\varphi_i^{\mathcal{K}}(Q, C) &= \sigma \alpha + (1 - \sigma)(c_i - \beta) = \\ &= \sigma \alpha - (1 - \sigma)\beta + (1 - \sigma)c_i = \\ &= \sigma \alpha - (1 - \sigma)\beta + \varphi_i^P(Q, C).\end{aligned}$$

Since, by equation (22), $\sigma \alpha - (1 - \sigma)\beta > 0$, it follows that

$$\varphi_i^{\mathcal{K}}(Q, C) > \varphi_i^P(Q, C);$$

and thus

$$r_i^{\mathcal{K}}(Q, C) = \frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} > \frac{\varphi_i^P(Q, C)}{c_i} = r_i^P(Q, C).$$

(2) $\beta \leq c_i \leq \alpha$. Then,

$$\varphi_i^{\mathcal{K}}(Q, C) = c_i - \beta(1 - \sigma) > c_i - \sigma c_i = \varphi_i^P(q, C).$$

Hence,

$$r_i^{\mathcal{K}}(Q, C) = \frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} > \frac{\varphi_i^P(Q, C)}{c_i} = r_i^P(Q, C).$$

Therefore, and summarizing, for each $i \in \mathcal{N}''$,

$$r_i^{\mathcal{K}}(Q, C) = \frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} > \frac{\varphi_i^P(Q, C)}{c_i} = r_i^P(Q, C). \quad (25)$$

Now, taking into account that, by definition, for any two agents i and j ,

$$r_i^P(Q, C) = \frac{\varphi_i^P(Q, C)}{c_i} = \frac{\varphi_j^P(q, C)}{c_j} = r_j^P(Q, C),$$

by equations (24) and (25), for any $i \in \mathcal{N}''$ and $j \in \mathcal{N}' \setminus \mathcal{N}''$,

$$r_i^{\mathcal{K}}(Q, C) = \frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} > \frac{\varphi_i^P(Q, C)}{c_i} = \frac{\varphi_j^P(Q, C)}{c_j} > \frac{\varphi_j^{\mathcal{K}}(Q, C)}{c_j} = r_j^{\mathcal{K}}(Q, C),$$

as established in statement (e) of Proposition 1.

To complete the proof, we proceed to explore for which set of agents the Compromise rule is progressive or regressive. We first prove that it is progressive in $\mathcal{N}'$ (statement (a) in Proposition 1).

Let us consider two agents i and j belonging to $\mathcal{N}'$, with $c_i \leq c_j$. Consider the following three cases, that exhaust all the possibilities.

(a) $\beta \leq c_i \leq c_j \leq \alpha$. Then,

$$\frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} = 1 - (1 - \sigma) \frac{\beta}{c_i} \leq 1 - (1 - \sigma) \frac{\beta}{c_j} = \frac{\varphi_j^{\mathcal{K}}(Q, C)}{c_j}.$$

(b) $c_i \leq \beta \leq c_j \leq \alpha$. Then,

$$\frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} = \sigma = 1 - (1 - \sigma) \leq 1 - (1 - \sigma) \frac{\beta}{c_j} = \frac{\varphi_j^{\mathcal{K}}(Q, C)}{c_j}.$$

(c) $c_i \leq c_j \leq \beta \leq \alpha$. Then,

$$\frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} = \sigma = \frac{\varphi_j^{\mathcal{K}}(Q, C)}{c_j}.$$

Therefore, as established in statement (a) of Proposition 1, the Compromise rule is progressive for agents in $\mathcal{N}'$.

To conclude the proof we just need to show that the Compromise rule is regressive for agents in $\mathcal{N}'' \setminus \mathcal{N}'$. Therefore, let us consider two agents i and j in $\mathcal{N}'' \setminus \mathcal{N}'$ such that $c_i \leq c_j$. Note that it must be the case that $\beta < \alpha < c_i \leq c_j$. This is because none of the agents is in $\mathcal{N}'$ and, from Alcalde and Peris (2017, Theorem 1), it follows that $\alpha > \beta$. Then, by equation (21),

$$\begin{aligned} \frac{\varphi_i^{\mathcal{K}}(Q, C)}{c_i} &= \frac{\alpha\sigma - \beta(1 - \sigma)}{c_i} + (1 - \sigma) \geq \\ &\geq \frac{\alpha\sigma - \beta(1 - \sigma)}{c_j} + (1 - \sigma) = \\ &= \frac{\varphi_j^{\mathcal{K}}(Q, C)}{c_j}, \end{aligned}$$

and thus, as established in statement (b) of Proposition 1, the rule $\varphi^{\mathcal{K}}$ is regressive in $\mathcal{N}'' \setminus \mathcal{N}'$. ■

Notice that the arguments provided in the previous proof of Proposition 1 can be adapted to demonstrate that the statements in Proposition 2 are true. For the latter result, it is important to take into account that, since $\sigma > 0.5$, then $\alpha < \beta$ (see Alcalde and Peris, 2017, Theorem 2).

B Proof of Proposition 3

Let us assume that (Q, C) satisfies condition (10). Then, for each agent i ,

$$\varphi_i^{UC}(Q, C) = \frac{c}{n} \leq c_i, \text{ and}$$

$$\varphi_i^{US}(Q, C) = c_i - \frac{S(Q, C)}{n} \geq 0.$$

Therefore,

$$\begin{aligned} \varphi_i^{\mathcal{K}}(Q, C) &= \sigma \varphi_i^{UC}(Q, C) + (1 - \sigma) \varphi_i^{US}(Q, C) = \\ &= \frac{S(Q, C)}{\sum_{j=1}^n c_j} \frac{c}{n} + \frac{c}{\sum_{j=1}^n c_j} \left[c_i - \frac{S(Q, C)}{n} \right] = \\ &= \frac{c}{\sum_{j=1}^n c_j} c = \varphi_i^P(Q, C). \end{aligned}$$

C Proof of Proposition 4

Note that $\sigma = 0.5$ for any problem (Q, C) satisfying that $c = S(q, C)$. Moreover, it is well-known that, in such a case, for each agent i ,

$$\varphi_i^P(Q, C) = \frac{c_i}{2},$$

and $\alpha = \beta$ (see Alcalde and Peris, 2017, Theorem 3). This allows to rewrite expression (21) as

$$\varphi_i^K(Q, C) = \begin{cases} \frac{\alpha}{2} + \frac{c_i - \beta}{2} & \text{if } \max\{\alpha, \beta\} < c_i \\ \frac{\alpha}{2} & \text{if } \alpha = \beta = c_i \\ \frac{c_i}{2} & \text{if } c_i < \min\{\alpha, \beta\} \end{cases} \quad (26)$$

Taking into account, again, that $\alpha = \beta$, equation (26) reduces to

$$\varphi_i^K(Q, C) = \frac{c_i}{2} = \varphi_i^P(Q, C).$$

■

D Proof of Theorem 1

From Propositions 3 and 4 we need to proof the only if part. Let us recall that, by equation (21),

$$\varphi_i^K(Q, C) = \begin{cases} \alpha\sigma + (1 - \sigma)(c_i - \beta) & \text{if } \max\{\alpha, \beta\} < c_i \\ \alpha\sigma & \text{if } \alpha \leq c_i \leq \beta \\ c_i - \beta(1 - \sigma) & \text{if } \beta \leq c_i \leq \alpha \\ c_i\sigma & \text{if } c_i < \min\{\alpha, \beta\} \end{cases}$$

Moreover, by Definition 1,

$$\varphi_i^P(Q, C) = (1 - \sigma)c_i. \quad (27)$$

Let us assume that, for each agent i , $\varphi_i^K(Q, C) = \varphi_i^P(Q, C)$. We also assume, without loss of generality, that $0 < c_1 \leq c_i$ for each $i > 1$.

Consider the following cases, that exhaust all the possibilities.

- (a) $c_1 < \min\{\alpha, \beta\}$. Then, since $\sigma c_1 = (1 - \sigma)c_1$, it follows that $\sigma = 1 - \sigma$, and thus $\sigma = 0.5$. That is, $c = S(Q, C)$.

(b) $\beta \leq c_1 \leq \alpha$. Then, since $\varphi_1^{\mathcal{K}}(Q, C) = c_1 - (1 - \sigma)\beta = (1 - \sigma)c_1 = \varphi_1^{\mathcal{P}}(Q, C)$,

$$\frac{\sigma}{1 - \sigma}c_1 = \beta \geq \frac{S(Q, C)}{n},$$

and thus,

$$c_1 \geq \frac{c}{n}, \quad \text{and} \quad c_1 \geq \beta \geq \frac{S(Q, C)}{n}.$$

(c) $\alpha \leq c_1 \leq \beta$. Then, since $\varphi_1^{\mathcal{K}}(Q, C) = \sigma\alpha = (1 - \sigma)c_1 = \varphi_1^{\mathcal{P}}(Q, C)$,

$$\frac{1 - \sigma}{\sigma}c_1 = \alpha \geq \frac{c}{n},$$

and thus,

$$c_1 \geq \frac{S(Q, C)}{n}, \quad \text{and} \quad c_1 \geq \alpha \geq \frac{c}{n}.$$

(d) $\max\{\alpha, \beta\} < c_1$. Then, by Definitions 2 and 3,

$$c_1 \geq \max\left\{\frac{S(Q, C)}{n}, \frac{c}{n}\right\}.$$

So, for each $i \in \mathcal{N}$,

$$nc_1 \geq \max\{S(Q, C), c\}.$$

■

E Proof of Theorem 2

Note that $\sigma < 0.5$ whenever $S(Q, C) < c$. Hence, $\alpha > \beta \geq \frac{S(Q, C)}{n}$. Therefore, for each $i \in \mathcal{N}$ such that $nc_i < S(Q, C)$,

$$c_i < \frac{S(Q, C)}{n} \leq \beta < \alpha.$$

Thus, from equation (21),

$$\varphi_i^{\mathcal{K}}(Q, C) = \sigma c_i < (1 - \sigma)c_i = \varphi_i^{\mathcal{P}}(Q, C).$$

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